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On a Nonlinear System of Partial Differential Equations

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We present in this paper some results concerning the following nonlinear system of P.D.E.

$$(\mathrm{P})\begin{cases} u'' - cv'' + \alpha D^4 u - M(a(u))D^2 u = f_1(x,t) & (1)\\ -cu'' + \gamma v'' + \delta D^4 v - \beta_0 D^2 v = f_2(x,t), & (2)\end{cases}$$

where $z' = z_t$, $Dz = z_x$ and $a(u) = \int_0^1 |Du|^2\, dx$.

The above system is a mathematical model which describes coupled flexural and torsional oscillations of an open cross-section beam. In Part I we consider the abstract initial value problem associated with the above system, prove the existence and uniqueness of solutions in a weak sense and mention two applications. In Part II we obtain regular solutions when adequate conditions on the data are assumed.

INTRODUCTION

A particular case of system (P), namely, when $M(\lambda) = m_0$ $(m_0 > 0)$ appears in Timoshenko *et al.* [13] as a mathematical model to describe coupled flexural and torsional vibrations of an open cross-section beam under certain conditions on longitudinal symmetry of the beam and on the load distribution.

The nonlinear term, which contains $\lambda = \int_0^L |Du|^2\, dx$, appears when the extensibility of the beam is taken into account (see, for example, Ball [1]). Dickey [3] assuming existence of solutions, in Fourier series form, solved a mixed problem associated with Eq. (1) alone, with homogeneous boundary conditions, in the case $M(\lambda) = m_0 + m_1\lambda$, $m_0 > 0$, $c = \alpha = 0$, $f_1 = 0$. Later, Dickey [4] considered the same equation but with $\alpha \neq 0$. The results of Dickey were generalized by Menzala [9] and Rivera [11]. A similar equation was also studied by Pohojaev [10]. Lions [6] studied the abstract problem associated with Eq. (1) in the case $c = \alpha = 0$ and $M = M(\lambda)$ of class $C^1(\mathbb{R}^+)$ and strictly positive. Medeiros [7] solved the same problem but

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admitted the nonlinear term of the form $(\alpha + M(\lambda))\, Du$ with $M \in C^1(\mathbb{R}^+)$, $M(\lambda) \geqslant m_0 + m_1 \lambda$, $\lambda \geqslant 0$, $\alpha \in R$ and obtained solutions in a Gevrey class.

Ball in [1] obtained weak and classical solutions of Eq. (1) in the case $c = 0$ and $M(\lambda) = m_0 + m_1 \lambda$, $m_0 > 0$. Brito [2] studied an abstract problem related to Eq. (1) when $c = 0$ and rotary inertia is taken into account.

We shall consider $M(\lambda) = m_0 + m_1 \lambda$, $m_1 > 0$. The results, however, are still valid if $M = M(\lambda) \in C^1(\mathbb{R}^+)$, $M(\lambda) \geqslant m_0 > 0$.

The coefficients in Eqs. (1) and (2) are constants and represent quantities which have to do with the physical model. In particular, the constant c represents the distance from the centroidal to the shear center axis of the beam (see Timoshenko *et al.* [13]). The two constants γ and c are related by $\gamma = I_p/S + c^2$, where S and I_p denotes the cross-section area and the polar moment of inertia, respectively, and so we have $\gamma > c^2$.

The Abstract Initial Value Problem

We consider, in this section, the abstract initial value problem associated with system P.

Let H be a Hilbert space with scalar product and norm denoted respectively by $(\cdot, \cdot)$ and $|\cdot|$. Let A be a linear map in H from domain $\mathrm{D}(A) \subset H$ into H. We shall also suppose that A is selfadjoint and positive. With the graph norm $\mathrm{D}(A)$ is a Hilbert space which we denote by V. It follows that V is continuously embedded in H. We shall also admit that the immersion of V in H is compact and that V is separable. We define A^2 by

$$\langle A^2 u, v \rangle = (Au, Av),$$

where V' is the dual of V and $\langle\ ,\ \rangle$ denotes the duality pairing $V'V$. It follows that $A^2 \in \mathscr{L}(V, V')$. If $u, v \in V$ let $a(u, v)$ and $b(u, v)$ be the bilinear forms given by

$$a(u, v) = (Au, v) \qquad \text{and} \qquad b(u, v) = (Au, Av).$$

If $A\colon \mathrm{D}(A) \subset H \to H$ satisfies the above conditions and u_0, v_0, u_1, v_1, f_1, f_2 are given functions we consider the problem of finding functions u and v solutions of the system

$$u'' + cv'' + \alpha A^2 u + (m_0 + m_1 |A^{1/2} u|^2) Au = f_1 \tag{3}$$

$$cu'' + \gamma v'' + \delta A^2 v + \beta_0 A v = f_2 \tag{4}$$

verifying the initial conditions

$$u(0) = u_0, \qquad v(0) = v_0; \qquad u'(0) = u_1, \qquad v'(0) = v_1. \tag{5}$$

THEOREM 1. *Let $u_0, v_0 \in V$, $u_1, v_1 \in H$, $f_1, f_2 \in L^2(0, T; H)$ be given. Then a unique pair of functions $\{u, v\}$ exists defined in $[0, T]$ $(0 < T < +\infty)$ with values in H such that:*

(1) $u, v \in L^\infty(0, T; V)$,

(2) $u', v' \in L^\infty(0, T; H)$.

The functions u, v are solutions of the system

$$(3) \quad \begin{aligned} &\frac{d}{dt}(u'(t), \omega) + c\frac{d}{dt}(v'(t), \omega) + \alpha(Au(t), A\omega) \\ &\qquad + (m_0 + m_1 a(u))(Au(t), \omega) = (f_1(t), \omega), \\ &c\frac{d}{dt}(u'(t), \omega) + \gamma\frac{d}{dt}(v'(t), \omega) + \delta(Av(t), A\omega) \\ &\qquad + \beta_0(Av(t), \omega) = (f_2(t), \omega), \end{aligned}$$

in the sense of $\mathscr{D}'(0, T)$, $\forall w \in V$.

(4) $u(0) = u_0$, $v(0) = v_0$; $u'(0) = u_1$, $v'(0) = v_1$.

Proof of Theorem 1 (a) Existence of solutions. Since we suppose V separable there are vectors $w_1, w_2, \dots, w_k$ in H such that the set $\{w_1, w_2, \dots, w_k\}$ is linearly independent for each $k \in N$ and the set of finite linear combinations of w_i are dense in V. We denote by V_m the subspace of V spanned by the first m vector $w_1, w_2, \dots, w_m$. We shall look for approximate solutions of system (3) of the form

$$u_m(t) = \sum_{j=1}^{m} g_{mj}(t) w_j; \qquad v_m(t) = \sum_{j=1}^{m} h_{mj}(t) w_j,$$

that is, $u_m, v_m \in V_m$.

The two functions u_m, v_m shall be determined so that they are solutions of the approximate system:

$$\begin{aligned} &(u_m''(t), w) + c(v_m''(t), w) + \alpha(Au_m(t), Aw) + (m_0 + m_1 a(u_m))(Au_m(t), w) \\ &\qquad = (f_1(t), w), \end{aligned} \tag{6}$$

$$\begin{aligned} &c(u_m''(t), w) + \gamma(v_m''(t), w) + \delta(Av_m(t), Aw) + \beta_0(Av_m(t), w) \\ &\qquad = (f_2(t), w), \end{aligned} \tag{7}$$

$\forall w \in V_m$ and satisfy the initial conditions

$$u_m(0) = u_{0m}, \qquad v_m(0) = v_{0m}; \qquad u_m'(0) = u_{1m}, \qquad v_m'(0) = v_{1m} \tag{8}$$

with

$$u_{0m} \to u_0, \qquad v_{0m} \to v_0 \qquad \text{strongly in} \quad V,$$
$$u_{1m} \to u_1, \qquad v_{1m} \to v_1 \qquad \text{strongly in} \quad H.$$

The system (6–8) of ordinary differential equations has unique solution, for each $m \in N$, in an interval $[0, t_m)$. The a priori estimates that follow show, in particular, that $t_m = T$, $0 < T < +\infty$. The method used to get these a priori estimates is standard in this type of problem so we omit the details.

Taking $w = 2u'_m(t)$ in Eq. (6) and $w = 2v'_m(t)$ in Eq. (7) we obtain

$$\frac{d}{dt}|u'_m|^2 + \alpha\frac{d}{dt}|Au_m|^2 + m_0\frac{d}{dt}|A^{1/2}u_m|^2 + \frac{m_1}{2}\frac{d}{dt}|A^{1/2}u_m|^4$$
$$= 2(f_1, u'_m) - 2c(v''_m, u'_m),$$
$$\gamma\frac{d}{dt}|v'_m|^2 + \delta\frac{d}{dt}|Av_m|^2 + \beta_0\frac{d}{dt}|A^{1/2}v_m|^2$$
$$= 2(f_2, v'_m) - 2c(u''_m, v'_m).$$

Adding the two equations and integrating in $(0, t)$, $t < t_m$, it follows that

$$|u'_m|^2 + \gamma|v'_m|^2 + \alpha|Au_m|^2 + \delta|Av_m|^2 + m_0|A^{1/2}u_m|^2$$
$$+\beta_0|A^{1/2}v_m|^2 + \tfrac{1}{2}m_1|A^{1/2}u_m|^4$$
$$\leqslant C + 2\int_0^t |(f_1, u'_m)|\,ds + 2\int_0^t |(f_2, v'_m)|\,ds + 2c\,|(u'_m, v'_m)|,$$

where C is a positive constant independent of m that bounds the terms involving the initial conditions. From now on C will denote a generic positive constant. Let $c_1 = \frac{1}{2}(1 - (c^2/\gamma))$ and $c_2 = \frac{1}{2}(\gamma - c^2)$. Then it follows from the last inequality that

$$c_1|u'_m|^2 + c_2|v'_m|^2 + \alpha|Au_m|^2 + \delta|Av_m|^2 + m_0|A^{1/2}u_m|^2 + \beta_0|A^{1/2}v_m|^2$$
$$\leqslant C + C\int_0^t \{|u'_m|^2 + |v'_m|^2\}\,ds.$$

By the Gronwall inequality we have

$$|u_m|, |v_m|, |u'_m|, |v'_m| \leqslant C, \qquad |Au_m|, |Av_m| \leqslant C, \qquad |A^{1/2}u_m|, |A^{1/2}v_m| \leqslant C, \tag{9}$$

where C is independent of m. So the solutions may, indeed, be extended to an interval $[0, T)$, $T < +\infty$. Moreover, the sequences (u_m), (v_m) belong to bounded sets in $L^{\infty}(0, T; V)$ and (u_m'), (v_m'), (Au_m), (Av_m), $(A^{1/2}u_m)$, $(A^{1/2}v_m)$ are in bounded sets in $L^{\infty}(0, T; H)$. Taking subsequences u_ν, v_ν it follows they converge weak-star to u and v, respectively, in $L^{\infty}(0, T; V)$ and u_ν', v_ν', Au_ν, Av_ν, $A^{1/2}u_\nu$, $A^{1/2}v_\nu$ converge weak-star to u', v', Au, Av, $A^{1/2}u$ and $A^{1/2}v$, respectively, in $L^{\infty}(0, T; H)$. By hypothesis the immersion of V in H is compact so it follows that u_ν converges strongly to u in $L^2(0, T; H)$ (Lions [5]).

As to the nonlinear term let $J(u_m) = a(u_m)Au_m$. Using the estimates we obtained, it results that $(J(u_m))$ belongs to a bounded set in $L^{\infty}(0, T; H)$. So we can take a subsequence $(J(v_\nu))$ that converges weak-star to a function ξ in $L^{\infty}(0, T; H)$. By monotonicity or by an argument used by Medeiros [8] or Ball [1] we obtain that $\xi = J(u) = a(u)Au$.

In (6) and (7) we write u_ν and v_ν instead of u_m and v_m and consider $\nu > m$, m now a fixed natural number. We pass to the limit as $\nu \to \infty$ and conclude that u, v satisfies condition (3) of the theorem. To verify the initial conditions we observe that $u, v \in L^{\infty}(0, T; v)$ and $u', v' \in L^{\infty}(0, T; H)$. Using the fact that u, v are solutions it can be shown that u'', $v'' \in L^{\infty}(0, T; V')$. So we can identify u and v with element of $C([0, T]; H)$ and u', v' with elements of $C([0, T]; V')$. Therefore $u(0)$, $v(0)$, $u'(0)$, $v'(0)$ are meaningful.

(b) Uniqueness. Suppose there are two pairs of solutions $\{u, v\}$ and $\{\hat{u}, \hat{v}\}$ and set $u = \bar{u} - \hat{u}$, $v = \bar{v} - \hat{v}$. Then u, v satisfies the system

$$u'' + cv'' + \alpha A^2 u + m_0 Au + m_1 [a(\bar{u})A\bar{u} - a(\hat{u})A\hat{u}] = 0 \tag{10}$$

$$cu'' + \gamma v'' + \delta A^2 v + \beta_0 Av = 0 \tag{11}$$

and initial conditions

$$u(0) = v(0) = 0; \qquad u'(0) = v'(0) = 0.$$

Following Lions [5] we define z and w by

$$z(t) = \int_t^s u(\sigma)\, d\sigma \ \text{ if } t \leqslant s \qquad \text{and} \qquad z(t) = 0 \qquad \text{if} \quad t > s,$$

$$w(t) = \int_t^s v(\sigma)\, d\sigma \ \text{ if } t \leqslant s \qquad \text{and} \quad w(t) = 0 \qquad \text{if} \quad t > s,$$

and set

$$z_1(t) = \int_0^t u(\sigma)\, d\sigma; \qquad w_1(t) = \int_0^t v(\sigma)\, d\sigma.$$

It follows that

$$\begin{aligned} z(t) &= z_1(s) - z_1(t), & w(t) &= w_1(s) - w_1(t), \\ z'(t) &= -u(t), & w'(t) &= -v(t), \\ z(s) &= 0, & w(s) &= 0, \\ z(0) &= z_1(s), & w(0) &= w_1(s). \end{aligned}$$

From (10) and (11) we have that

$$\int_0^s \{\langle u'', z\rangle + c\langle v'', z\rangle + \alpha(Au, Az) + m_0(Au, z) + m_1(a(\bar{u})A\bar{u} - a(\hat{u})A\hat{u}, z\}\, d\sigma = 0$$

$$\int_0^s \{c\langle u'', w\rangle + \gamma\langle v'', w\rangle + \delta(Av, Aw) + \beta_0(Av, w)\}\, d\sigma = 0.$$

From these two relations we obtain, after some algebraic calculations,

$$|u(s)|^2 + \gamma\,|v(s)|^2 + \alpha\,|Az_1(s)|^2 + m_0 a(z_1(s)) + \beta_0 a(w_1(s)) \leqslant 2c\,|u(s)|\,|v(s)| + 2m_1 \int_0^s |(a(\bar{u})A\bar{u} - a(\hat{u})A\hat{u}, z(\sigma))|\, d\sigma.$$

Using estimates (9) for $\bar{u}$ and $\hat{u}$ it follows that the integrand of the integral of the last inequality is bounded by $C\,|u|^2 + C\,|Az|^2$. Also, from the properties of $z = z(t)$ we have

$$|Az(\sigma)|^2 \leqslant 2\{|Az_1(s)|^2 + |Az_1(\sigma)|^2\},$$

and the last inequality becomes

$$C_1\,|u(s)|^2 + C_2\,|v(s)|^2 + (\alpha - 2Cs)\,|Az_1(s)|^2 + m_0 a(z_1(s)) + \beta_0 a(w_1(s)) \leqslant 2C \int_0^s \{|u(\sigma)|^2 + |Az_1(\sigma)|^2\}\, d\sigma.$$

Given ε_0, $0 < \varepsilon_0 < \alpha$, let us choose s_0 such that $s_0 > 0$ and $\alpha - 2Cs_0 > \varepsilon_0$. Therefore, for all $s \in [0, s_0]$ we have $\alpha - 2Cs > \alpha - 2Cs_0 > \varepsilon_0$. With s_0 so chosen it follows that

$$|u(s)|^2 + |v(s)|^2 + \varepsilon_0\,|Az_1(s)|^2 \leqslant C \int_0^s \{|u(\sigma)|^2 + |Az_1(\sigma)|^2\}\, d\sigma.$$

This implies, by the Gronwall inequality, that $u(s) = 0$ and $v(s) = 0$ for all $s \in [0, s_0]$. By a standard argument we conclude that $u = 0$, $v = 0$ in $[0, T]$.

Applications

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with regular boundary $\partial\Omega$, $A = -\Delta$ and $a(u) = |\nabla u|^2$.

(a) Let $H = L^2(\Omega)$, $V = H_0^2(\Omega)$, $V' = H^{-2}(\Omega)$. If $u_0, v_0 \in V$, u_1, $v_1 \in H, f_1, f_2 \in L^2(0, T; H)$ are given functions, then there is a unique pair of functions $u, v \in L^\infty(0, T; V)$, u', $v' \in L^\infty(0, T; H)$ and

$$u'' - cv'' + \alpha\Delta^2 u - \left(m_0 + m_1 \int_\Omega |\nabla u|^2 \, dx\right) \Delta u = f_1$$

$$cu'' + \gamma v'' + \delta\Delta^2 v - \beta_0 \Delta v = f_2 \tag{13}$$

$$u(0) = u_0, \qquad v(0) = v_0; \qquad u'(0) = u_1, \qquad v'(0) = v_1$$

When $n = 1$, $\Omega = (0, L)$, $A = -\partial^2/\partial x^2$ this corresponds to the case of a beam with fixed end points which cannot rotate.

(b) Now let $H = L^2(\Omega)$, $V = H_0^1(\Omega) \cap H^2(\Omega)$. Hypothesis on A are satisfied. Let $u_0, v_0 \in V$, $u_1, v_1 \in H, f_1, f_2 \in L^2(0, T; H)$. Then there exists a unique pair of functions u, v solutions of problem (13).

In the case $n = 1$, $\Omega = (0, L)$, $A = -\partial^2/\partial x^2$ this corresponds to a beam in which the linear displacement and the bending moments are zero at the end points. (See Timoshenko [12]).

Regular Solutions

Let us consider again the system (3) – (4) with initial conditions (5) and suppose now that A is a linear operator in H, with dense domain $\mathrm{D}(A)$, A selfadjoint and positive. We suppose also that there are sequences of numbers $\{\lambda_i\}_{i\in\mathbb{N}}$ and vectors $\{\omega_i\}_{i\in\mathbb{N}}$ such that $A\omega_j = \lambda_j\omega_j$, $j \in \mathbb{N}$, $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to \infty$ and the set of eigenvectors is dense in H.

Now $u(t)$, $v(t)$ are vectors of H, t a real variable, and the derivative of $u(v)$ with respect to the norm of H denoted by u' (v') or du/dt (dv/dt). As before $(\cdot, \cdot)$ and $|\cdot|$ denotes the scalar product and norm in H. We define $a(u, v) = (Au, v)$ and $a(u) = (Au, u)$, $u \in \mathrm{D}(A)$.

Regular solutions are shown to exist according to the following theorem.

THEOREM 2. *Suppose that the operator A satisfies the above conditions and that u_0, v_0, u_1, v_1, f_1, f_2 are given functions such that*

(1) $u_0, v_0 \in \mathrm{D}(A^{k+1/2})$

(2) $u_1, v_1 \in \mathrm{D}(A^k)$

(3) $f_1, f_2 \in \mathrm{D}(A^k)$; $A^k f_1$, $A^k f_2 \in L^2(0, T; H)$ *for all* $k = 0, 1, 2, \ldots$

Then there is a unique pair of functions u, v from $[0, T] \to H$, $0 < T < +\infty$ satisfying the conditions:

(4) $u(t), v(t), u'(t), v'(t) \in D(A^k)$

(5) $u, v, u', v' \in C^0([0, T]; D(A^k))$ *for all* $k = 0, 1, 2, \ldots$.

(6) u, v *satisfies system* (3)–(4) *and initial conditions* (5).

Proof. (a) Existence of Regular Solutions. Let $V_m = [w_1, w_2, \ldots, w_m]$ be the subspace of H generated by the m first eigenvectors of A and let u_m, v_m be defined by $u_m(t) = \sum_{i=1}^m g_{im}(t) w_i$, $v_m(t) = \sum_{i=1}^m h_{im}(t) w_i$. Functions u_m, $v_m \in V_m$ are to be solutions of the approximate system

$$(u''_m(t), w) + c(v''_m(t), w) + \alpha(Au_m(t), Aw)$$

$$+ (m_0 + m_1 a(u_m(t)))(Au_m(t), w) = (f_1, w) \tag{6'}$$

$$c(u''_m(t), w) + \gamma(v''_m(t), w) + \delta(Av_m(t), Aw) + \beta_0(Av_m(t), w) = (f_2, w), \tag{7'}$$

$\forall w \in V_m$ and satisfying the initial conditions

$$u_m(0) = u_{0m}, \qquad v_m(0) = v_{0m}; \qquad u'_m(0) = u_{1m}, \qquad v'_m(0) = v_{1m}, \tag{8'}$$

where

$$u_{\alpha m} = \sum_{i=1}^{m} (u_\alpha, w_i) w_i, \qquad \alpha = 0, 1.$$

This system of ordinary differential equations has unique solutions u_m, v_m in an interval $(0, t_m)$, for each $m \in \mathbb{N}$.

Estimates 1

Proceeding exactly in the same way as we did in Part 1 we get the a priori estimates (9) for u_m, v_m, u'_m, v'_m, Au_m, Av_m, $A^{1/2}u_m$, and $A^{1/2}v_m$. These estimates permit us to extend u_m, v_m to the interval $[0, T]$, $T < +\infty$, T independent of m.

Estimates 2

Now let $w = 2A^{2k}u'_m$ in (6′) and $w = 2A^{2k}v'_m$ in (7′). Taking into account the properties of A, the hypothesis on the initial data and on f_1, f_2 we obtain

$$\frac{d}{dt}\{|A^k u'_m|^2 + |A^{k+1}u_m|^2 + [m_0 + m_1 a(u_m)]\,|A^{k+1/2}u_m|^2\}$$
$$= 2(A^k f_1, A^k u'_m) - 2c(A^k u''_m, A^k u'_m) + 2m_1 |A^{k+1/2}u_m|^2 \frac{d}{dt} a(u_m).$$

$$\frac{d}{dt}\{\gamma\,|A^k v'_m|^2 + \delta\,|A^{k+1}v_m|^2 + \beta_0\,|A^{k+1/2}v_m|^2\}$$
$$= 2(A^k f_2, A^k v'_m) - 2c(A^k u''_m, A^k v'_m).$$

Adding the two equations, integrating and bounding the terms on the second member the following inequality is obtained:

$$c_1 |A^k u'_m|^2 + c_2 |A^k v'_m|^2 + \alpha |A^{k+1} u_m|^2 + \delta |A^{k+1} v_m|^2 + m_0 |A^{k+1/2} u_m|^2 + \beta_0 |A^{k+1/2} v_m|^2 \leqslant C \int_0^t \{|A^k u'_m|^2 + |A^{k+1/2} u_m|^2\} \, ds, \tag{14}$$

where, as before, $c_1 = \frac{1}{2}(1 - (c^2/\gamma))$, $c_2 = \frac{1}{2}(\gamma - c^2)$.

From (14) and the Gronwall inequality it follows that

$$|A^k u'_m|^2 + |A^k v'_m|^2 + |A^{k+1} u_m|^2 + |A^{k+1} v_m|^2 + |A^{k+1/2} u_m|^2 + |A^{k+1/2} v_m|^2 \leqslant C \tag{15}$$

in $(0, T)$, C independent of m.

Limit of Approximate Solutions

We shall prove first that the sequence $(A^k u'_m)$, $(A^k v'_m)$, $(A^{k+1/2} u_m)$, $(A^{k+1/2} v_m)$, $(A^{k+1} u_m)$, $(A^{k+1} v_m)$ are Cauchy sequences in $[0, T]$, $T < +\infty$. Suppose that u_{n_2}, u_{n_1}, v_{n_2}, v_{n_1} are approximate solutions of system (6′), (7′) with initial conditions (8′). Let n_2, n_1 be natural numbers and $n_2 > n_1$. Set $w = u_{n_2} - u_{n_1}$ and $z = v_{n_2} - v_{n_1}$. Then for all $\eta \in V_m$ $(m \geqslant n_2)$ w, z satisfies the following two relations:

$$(w'', \eta) + c(z'', \eta) + \alpha(Aw, A\eta) + m_0(A\omega, \eta) + m_1 a(u_{n_1})(A\omega, \eta) + m_1[a(u_{n_2}) - a(u_{n_1})]\, a(u_{n_2}, \eta) = 0 \tag{16}$$

$$c(w'', \eta) + \gamma(z'', \eta) + \delta(Az, A\eta) + \beta_0(Az, \eta) = 0. \tag{17}$$

Taking $\eta = 2A^{2k} w'$ in (16) and $\eta = 2A^{2k} z'$ in (17). It follows that

$$|A^k w'|^2 + \gamma |A^k z'|^2 + \alpha |A^{k+1} w|^2 + \delta |A^{k+1} z|^2 + m_0 |A^{k+1/2} w|^2 + \beta_0 |A^{k+1/2} z|^2 \leqslant |K_k(0)| + 2c |A^k w'| \, |A^k z'| + 2m_1 C \int_0^t \{|A^{k+1/2} w|^2 + |A^{1/2} w|^2 + |A^k w'|^2\} \, ds, \tag{18}$$

where

$$K_k(0) = |A^k w'(0)|^2 + \gamma |A^k z'(0)|^2 + \alpha |A^{k+1} w(0)|^2 + \delta |A^{k+1} z(0)|^2 + (m_0 + m_1 a(u_{n_2}(0))) |A^{k+1/2} w(0)|^2 + \beta_0 |A^{k+1/2} z(0)|^2 + 2c(A^k w'(0), A^k z'(0)).$$

The hypothesis on the initial data imply that $K_k(0) \to 0$, as $n_2, n_1 \to \infty$, for all $k = 0, 1, 2, \ldots$.

For $k = 0$ in (18) it follows that

$$|w'|^2 + |z'|^2 + |Aw|^2 + |Az|^2 + |A^{1/2}w|^2 + |A^{1/2}z|^2$$
$$\leqslant C_0 K_0(0) + C \int_0^t \{|A^{1/2}w|^2 + |w'|^2\}\, ds$$

hence by the Gronwall inequality

$$|A^{1/2}w|^2 \leqslant C_0 K_0(0) e^{CT}, \qquad T < +\infty. \tag{19}$$

Taking (19) into (18) it follows that

$$|A^k w'|^2 + |A^k z'|^2 + |A^{k+1}w|^2 + |A^{k+1}z|^2 + |A^{k+1/2}w| + |A^{k+1/2}z|^2$$
$$\leqslant I_{0k}(T) + C \int_0^t \{|A^k w'|^2 + |A^{k+1/2}w|^2\}\, ds,$$

where

$$I_{0k}(T) = C_0 K_k(0) + CC_0 K_0(0)\, Te^{CT}, \qquad T < +\infty.$$

The last inequality implies that

$$|A^k w'|^2 + |A^k z'|^2 + |A^{k+1}w|^2 + |A^{k+1}z|^2 + |A^{k+1/2}w| + |A^{k+1/2}z|^2$$
$$\leqslant I_{0k}(T) e^{CT}, \qquad \forall t \in [0, T].$$

But $\lim_{n_2, n_1 \to \infty} I_{0k}(T) = 0$, $\forall k \in \mathbb{N}$, so the sequences $(A^k u_n'(t))$, $(A^k v_n'(t))$, $(A^{k+1}u_n(t))$, $(A^{k+1}v_n(t))$, $(A^{k+1/2}u_n(t))$, $(A^{k+1/2}v_n(t))$ are uniform Cauchy sequences of $C^0([0, T], H)$. Therefore they are convergent for all $k \in N$. Thus for $k = 0$ we have $u = u(t)$, $v = v(t)$ such that $(u_n(t))$, $(v_n(t))$ converge uniformly to $u(t)$, $v(t)$ in $C^0([0, T], H)$ and

$$u, v; u', v' \in C^0([0, T], D(A^k)) \qquad \text{for all} \quad k = 0, 1, 2, \ldots.$$

We also have the uniform convergence of $(A^{1/2}u_n)$ to $(A^{1/2}u)$ in $C^0([0, T]; H)$ and hence we may pass to the limit in the nonlinear term. It follows from the approximate problem that

$$\frac{d}{dt}(u'(t), w) + c\frac{d}{dt}(v'(t), w) + \alpha(Au(t), Aw)$$
$$+ (m_0 + m_1 a(u(t)))(Au(t), w) = (f_1, w)$$
$$c\frac{d}{dt}(u'(t), w) + \gamma\frac{d}{dt}(v'(t), w) + \delta(Av(t), Aw) + \beta_0(Av(t), w)$$
$$= (f_2, w), \qquad \forall w \in H.$$

The uniform convergence of $(u_n(t))$, $(v_n(t))$, $(u_n'(t))$, $(v_n'(t))$ in $C^0([0, T]; H)$ guarantees that the initial conditions are satisfied.

Uniqueness

Suppose there are two pairs of regular solutions $\{u_1, v_1\}$, $\{u_2, v_2\}$ in $C^0([0, T]; H)$ of the system (3), (4) satisfying initial conditions (5). Let $w = u_2 - u_1$ and $z = v_2 - v_1$ and $v_2 - v_1$. Then $\{w, z\}$ satisfies the following two equations:

$$w'' + cz'' + \alpha A^2 w + m_0 A w + m_1 a(u_2) A w = m_1 [a(u_1) - a(u_2)] A u_1$$
$$cw'' + \gamma z'' + \delta A^2 z + \beta_0 A z = 0$$

and the initial conditions

$$w(0) = z(0) = 0; \qquad w'(0) = z'(0) = 0.$$

Taking the scalar product of the first equation with $2w'$ and of the second one with $2z'$, integrating in $[0, t]$, $t \leqslant T$, we get

$$\begin{aligned} &|w'|^2 + \gamma |z'|^2 + \alpha |Aw|^2 + |Az|^2 + m_0 |A^{1/2} w|^2 + \beta_0 |A^{1/2} z|^2 \\ &\leqslant 2c |w'| |z'| + 2m_1 \int_0^t |A^{1/2} w|^2 \left| \frac{d}{ds} a(u_2) \right| ds \\ &+ 2m_1 \int_0^t [a(u_2) - a(u_1)] |Au_1| |w'| \, ds. \end{aligned}$$

From this inequality and the estimates we obtained before, it follows that

$$\begin{aligned} &|w'|^2 + |z'| + |Aw|^2 + |Az|^2 + |A^{1/2} w|^2 + |A^{1/2} z|^2 \\ &\leqslant C \int_0^t \{|w'|^2 + |Aw|^2 + |A^{1/2} w|^2\} \, ds \end{aligned}$$

and this implies that $w = 0$, $z = 0$ in $[0, T]$.

Applications

Let H be the Hilbert space $L^2(\Omega)$ for $\Omega \subset \mathbb{R}^n$ an open bounded set with regular boundary $\partial\Omega$. We take $A = -\Delta$, where Δ is the Laplacian operator, and consider the system:

$$\begin{aligned} u'' + cv'' + \alpha \Delta^2 u - \left(m_0 + m_1 \int_\Omega |\nabla u|^2 \, dx \right) \Delta u &= f_1 \\ cu'' + \gamma v'' + \delta \Delta^2 v - \beta_0 \Delta v &= f_2 \end{aligned} \tag{20}$$

together with the initial conditions

$$u(0)=u_0, \qquad v(0)=v_0; \qquad u'(0)=u_1, \qquad v'(0)=v_1. \tag{21}$$

(a) We take $u_0, v_0 \in D(A^{k+1/2})$, $u_1, v_1 \in D(A^k)$ and $D(A) = H^4(\Omega) \cap H_0^2(\Omega)$.

Then $\{u, v\}$ is the unique solution of (2) satisfying initial conditions (21), $u, v \in C^0([0, T]; D(A^k))$.

(b) Let $u_0, v_0 \in D(A^{k+1/2})$, $u_1, v_1 \in D(A^k)$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Again, $\{u, v\}$ is the unique solution of (20) satisfying (21), $u, v \in C^0([0, T]; D(A^k))$. When $n = 1$ the interpretations are identical with those given in Part 1.

We note again that these results remain valid if the nonlinear terms has for coefficient a function $M = M(\lambda)$, $\lambda \geqslant 0$, $M \in C^1$, and $M \geqslant m_0 > 0$.

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